CHARACTERIZATIONS OF TAME SURFACES IN E^3

BY

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1. **Introduction.** Various conditions under which a surface is tame in E^3 have been given by Bing [6], [7], Griffith [17], Harrold [18], and Hempel [19]. Examples of some types of wild spheres have been given by Antoine [2], Alexander [1], Fox and Artin [15], and Bing [8]. Using Bing's theorem that a surface is tame if its complement is 1-ULC [7], we show in §3 that a connected 2-manifold K is tame in E^3 if it can be locally spanned, with a disk, at each point from each side of K, and in §6 this is generalized to a 2-manifold in a 3-manifold. This characterization of tame surfaces includes Harrold's theorem [18] that a surface is tame in E^3 if it is locally peripherally unknotted, but we neither require that the boundary of a spanning disk should be tame nor that at each point of K there should be two spanning disks that have the same boundary and are on opposite sides of K. This enables us to give conditions under which a surface is tame from one side. There exist wild spheres of the types described in the above references that are tame from one complementary domain.

Most of the definitions used here will be like those used by Bing in [7]. A subset Y of a metric space is 1-ULC (*uniformly locally simply connected*) if for each $\epsilon > 0$ there is a $\delta > 0$ such that each map f of the boundary of a disk K into a δ-subset(2) of Y can be shrunk to a point in an ϵ-subset of Y; that is, f can be extended to a map of K into an ϵ-subset of Y. The set Y is *locally simply connected* at a point p of the closure of Y if for each neighborhood N of p there is an open set U containing p such that each map of a simple closed curve into $U \cdot Y$ can be shrunk to a point in $N \cdot Y$.

A 2-manifold M is *tame* in E^3 if there is a homeomorphism of E^3 onto itself that carries M onto a polyhedron. If V is a complementary domain of a 2-manifold M in E^3 such that $M + V$ is a 3-manifold with boundary, we say that M is *tame from* V. This definition is reasonable in view of the fact that any 3-manifold with boundary can be triangulated [3], [5], [21]. It follows from these triangulation results by Bing and Moise, and

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(2) We use the prefix in "δ-subset," "δ-disk," etc., to imply that a set has a diameter less than δ. We use the prefix in "2-sphere," "2-manifold," and "3-manifold" to imply the dimension, but there should be no confusion between the two usages.

also from more recent results by Brown [13], that a 2-manifold M in E^3 is tame from the complementary domain V of M if and only if M can be collared in V. In this context, a connected 2-manifold M in E^3 is tame if and only if M is tame from each of its complementary domains. Except in §6, we restrict our discussion to connected 2-manifolds in E^3 and use the fact that each such manifold has exactly two complementary domains and is the boundary of each of them. Some characterizations of tame 2-manifolds in E^3, without any requirement that the manifolds be connected, are included in §6.

We define a 2-manifold M in E^3 to be *locally tame* at a point p from a complementary domain V of M if p lies in a subset U of $M + V$ such that U is open relative to $M + V$ and the closure of U is a topological cube. This is equivalent to saying that M is locally collared in V at p [13].

For a 2-sphere S in E^3, we will use $\operatorname{Int} S$ and $\operatorname{Ext} S$ to denote the bounded complementary domain and the unbounded complementary domain, respectively, of S. For a disk K, we will use $\operatorname{Int} K$ to denote $K - \operatorname{Bd} K$.

2. **Surfaces that are tame from one side.** The main purpose of this section is to identify some conditions under which a connected 2-manifold M in E^3 is tame from one of its complementary domains, and these results will be used to establish the characterization of tame surfaces in the next section. Bing [6], [7] has already established Theorems 4, 5, and 6 for the case where M is a 2-sphere and for the case where M is a connected 2-manifold each of whose complementary domains satisfies the hypotheses of these theorems. We use his results and adapt his methods to give conditions that imply that a connected 2-manifold is tame from one of its complementary domains. While the first three parts of the conclusion of Theorem 1 have already been established by Bing [7], we need the additional two parts in the proofs of subsequent theorems.

THEOREM 1. *Suppose M is a connected 2-manifold in E^3, p is a point of M, and V is a complementary domain of M. Then for each $\epsilon > 0$ there is a disk D in M and a 2-sphere S such that*

(1) $\operatorname{diam} S < \epsilon$,

(2) $p \in \operatorname{Int} D \subset S$,

(3) *S is locally polyhedral at each point of $S - D$,*

(4) *$\operatorname{Bd} D$ is tame, and*

(5) *D is on the boundary of $V \cdot \operatorname{Int} S$.*

Proof. Let E be a disk in M such that $p \in \operatorname{Int} E$, and let C be a cube of diameter less than ϵ such that $p \in \operatorname{Int} C$ and $\operatorname{Bd} E \subset E^3 - C$. It follows from Bing's Theorems 1 and 8 in [9] that there is a disk D in $M \cdot \operatorname{Int} C$ such that $p \in \operatorname{Int} D$ and $\operatorname{Bd} D$ is tame. There exists an arc pq such that $pq - q \subset \operatorname{Int} C$, $q \in \operatorname{Bd} C$, $pq \cdot E = p$, and p is not a limit point of $pq \cdot V$.

There exists a topological ray X which contains pq and is unbounded in E^3 such that $X - pq \subset E^3 - C$. Now a sphere S satisfying the requirements of the conclusion of Theorem 1 can be obtained by following Bing's procedure [7, Theorem 5] in obtaining $h(E)$ and E' provided it is required that $h(E)$ should not intersect X and that the polygonal disks near $\text{Bd}\, C$ that are added to E' should not intersect X.

LEMMA 1. *If $D_1, D_2, \cdots, D_n$ are disjoint disks in E^3 and f is a map of a disk K into E^3 such that $f(\text{Bd}\, K) \subset E^3 - \sum_{i=1}^n D_i$, then there is a map g of K into E^3 such that*

(1) *$g|\text{Bd}\, K = f|\text{Bd}\, K$,*

(2) *$g(K) \subset f(K) + \sum_{i=1}^n \text{Int}\, D_i$, and*

(3) *$g(K) - \sum_{i=1}^n D_i$ is connected.*

Proof. For each i, there exists a disk D_i' such that $D_i' \subset D_i$, $D_i' \cdot f(K) = D_i \cdot f(K)$, and $\text{Bd}\, D_i' \cdot \text{Bd}\, D_i = f(K) \cdot \text{Bd}\, D_i$. Let H denote the component of $K - f^{-1}(f(K) \cdot \sum_{i=1}^n D_i')$ that contains $\text{Bd}\, K$. For each i $(1 \leqq i \leqq n)$, let C_i denote the set of all limit points of H that are carried by f into D_i', and let B_i denote the sum of all components of $K - H$ that intersect C_i. It follows that $B_1, B_2, \cdots, B_n$ are disjoint and that $H + \sum_{i=1}^n B_i = K$. By the Tietze Extension Theorem [20, p. 82], there exists for each i $(1 \leqq i \leqq n)$ a map f_i of B_i into D_i' so that $f_i | C_i = f | C_i$. Now a map g such that $g|H = f|H$ and $g|B_i = f_i$ satisfies the requirements in the conclusion of Lemma 1.

LEMMA 2. *Suppose S is a 2-sphere in E^3, X is a nondegenerate subcontinuum of S, and for each $\gamma > 0$ there exists a 2-sphere S' such that $X \subset \text{Ext}\, S'$ and S' is homeomorphically within γ of S. Then for each $\epsilon > 0$, there exists a $\delta > 0$ such that each map of a simple closed curve into a δ-subset of $\text{Int}\, S$ can be shrunk to a point in an ϵ-subset of $\text{Int}\, S + S - X$.*

Proof. Let ϵ be a positive number such that

$$\epsilon < \text{diam}\, X. \tag{1}$$

There exist positive numbers α and σ such that

$$3\alpha < \epsilon, \tag{2}$$

(3) every α-subset of S lies in an $\epsilon/3$-disk on S, and

(4) every σ-subset of S lies in an $\alpha/3$-disk on S.

Let δ be a positive number such that

$$2\delta < \sigma \quad \text{and} \quad 3\delta < \alpha. \tag{5}$$

We will show that δ satisfies the requirements in the conclusion of Lemma 2.

Let K be a disk and let f be a map of $\text{Bd}\, K$ into a δ-subset of $\text{Int}\, S$. Then f can be extended to map K into a δ-subset of E^3. Let γ be a positive number such that

(6) $$4\gamma < \sigma$$

and

(7) $$6\gamma < \alpha.$$

By hypothesis, there exists a 2-sphere S' such that S' is homeomorphically within γ of S and $X \subset \operatorname{Ext} S'$. We can impose the requirement that $f(\operatorname{Bd} K) \subset \operatorname{Int} S'$ [20, p. 97]. From (4) it follows that

(8) every $(\sigma - 2\gamma)$-subset of S' lies in an $(\alpha/3 + 2\gamma)$-disk on S'.

Since $\operatorname{diam} f(K) < \delta$, it follows from (5) that

(9) $$\operatorname{diam}[f(K) \cdot S'] < \sigma/2.$$

Now (6) and (9) combine to give

(10) $$\operatorname{diam}[f(K) \cdot S'] < \sigma - 2\gamma,$$

and (8) and (10) imply that $f(K) \cdot S'$ lies in an $(\alpha/3 + 2\gamma)$-disk on S'. By (7), $\alpha/3 + 2\gamma < 2\alpha/3$, so that $f(K) \cdot S'$ lies in a $2\alpha/3$-disk D' on S'. Lemma 1 implies that there exists a map f' of K into $D' + \operatorname{Int} S'$ such that $f' | \operatorname{Bd} K = f | \operatorname{Bd} K$ and $f'(K) \subset f(K) + D'$. Since $\operatorname{diam} f(K) < \delta$ and $\operatorname{diam} D' < 2\alpha/3$, it follows from (5) that

(11) $$\operatorname{diam} f'(K) < \alpha.$$

Now $f'(K) \cdot S$ is a subset of $S - X$, so (3) implies that $f'(K) \cdot S$ lies in an $\epsilon/3$-disk on S. Hence it follows from (1) that $f'(K) \cdot S$ can be covered by a finite number of disjoint $\epsilon/3$-disks $D_1, D_2, \cdots, D_n$ each lying in $S - X$. Now apply Lemma 1 to get a map g of K into $\operatorname{Int} S + \sum_{i=1}^n D_i$ such that $g | \operatorname{Bd} K = f' | \operatorname{Bd} K$ and $g(K) \subset f'(K) + \sum_{i=1}^n D_i$. By combining (2) and (11) with the fact that each D_i is an $\epsilon/3$-disk, we conclude that

$$\operatorname{diam} g(K) < \alpha + 2\epsilon/3 < \epsilon.$$

Thus we have shown that $f(\operatorname{Bd} K)$ can be shrunk to a point in an ϵ-subset of $\operatorname{Ind} S + S - X$.

LEMMA 3. *If S is a 2-sphere in E^3 and J is a tame simple closed curve on S, then for each $\epsilon < 0$ there exists a $\delta > 0$ such that each map of a simple closed curve into a δ-subset of $\operatorname{Int} S$ can be shrunk to a point in an ϵ-subset of $\operatorname{Int} S + S - J$.*

Proof. We will show that J has the properties required of X in the hypothesis of Lemma 2. Let γ be a positive number and let D_1 and D_2 be the two disks on S which have J as a boundary. It follows from Bing's approximation theorem for surfaces [4], [10] that there exist two disks D_1' and D_2' each having J as a boundary such that (1) D_1' and D_2' are locally polyhedral except on J and (2) $D_1' + D_2'$ is a 2-sphere S'' which is homeomorphically within $\gamma/2$ of S. Since S'' is locally tame at each point of $S''-J$, it follows that S'' is tame [14]. Hence there is a 2-sphere S' which is in $\operatorname{Int} S''$ and which is homeomorphically within $\gamma/2$ of S''. It follows that $J \subset \operatorname{Ext} S'$ and S' is homeomorphically within γ of S. The conclusion of Lemma 3 now follows from Lemma 2.

LEMMA 4. *Suppose S is a 2-sphere in E^3 and f is a map of a disk K into $S+\operatorname{Int} S$ such that $f(\operatorname{Bd} K) \subset \operatorname{Int} S$ and $\operatorname{Int} S$ is locally simply connected at each point of $S\cdot f(K)$. Then for each $\epsilon>0$, $f(\operatorname{Bd} K)$ can be shrunk to a point in $\operatorname{Int} S$ in an ϵ-neighborhood of $f(K)$.*

Proof. From the hypothesis that $\operatorname{Int} S$ is locally simply connected at each point of $S\cdot f(K)$, it follows that $S\cdot f(K)$ can be covered by a finite number of open sets $U_1, U_2, \cdots, U_n$ so that U_i $(1\leqq i\leqq n)$ does not intersect $f(\operatorname{Bd} K)$ and each map of a simple closed curve in $U_i\cdot \operatorname{Int} S$ can be shrunk to a point in an $\epsilon/2$-subset of $\operatorname{Int} S$. There is a positive number σ such that every σ-subset of $S+\operatorname{Int} S$ that intersects $S\cdot f(K)$ is in some U_i. Also, there exists a positive number α so that $\alpha<\epsilon$, $\alpha<\sigma/2$, and each two points in $\operatorname{Int} S$ that are a distance apart less than α can be joined by an arc in $\operatorname{Int} S$ of diameter less than $\sigma/6$ [24, p. 66].

Now let T be a triangulation of K such that each simplex in T has an image under f that is of diameter less than $\alpha/3$. Let T' denote the set of all 2-simplexes in T that have an image, under f, which intersects S. Let t denote some element of T', and let p_1, p_2, and p_3 denote the vertices of t. If $f(p_i)\in \operatorname{Int} S$, let $q_i=f(p_i)$. If $f(p_i)\in S$, let q_i be a point of $\operatorname{Int} S$ that is within a distance $\alpha/6$ of $f(p_i)$. Furthermore, if p_i is a vertex shared by some two or more elements of T, then we choose the same point q_i for all of these simplexes. Each two of the points q_1, q_2, and q_3 are within a distance α of each other, so there exist singular arcs(3) q_1q_2, q_2q_3, and q_3q_1, each of which lies in $\operatorname{Int} S$ and has a diameter less than $\sigma/6$. In case $f(p_ip_j)\subset \operatorname{Int} S$, where $i\neq j$ and p_ip_j is the 1-simplex in T with vertices p_i and p_j, we let $q_iq_j=f(p_ip_j)$. Hence $q_1q_2+q_2q_3+q_3q_1$ has a diameter less than $\sigma/2$ and lies in a σ-subset of $S+\operatorname{Int} S$ that intersects $S\cdot f(K)$. Thus $q_1q_2+q_2q_3+q_3q_1$ lies in some U_i, so this map of a simple closed curve can be shrunk to a point in an $\epsilon/2$-subset of $\operatorname{Int} S$ in an ϵ-neighborhood of $f(K)$. By following such a procedure for each 2-simplex in T', $f(\operatorname{Bd} K)$ can be shrunk to a point in $\operatorname{Int} S$ in an ϵ-neighborhood of $f(K)$.

THEOREM 2. *If S is a 2-sphere in E^3, J is a tame simple closed curve on S, and $\operatorname{Int} S$ is locally simply connected at each point of $S-J$, then S is tame from $\operatorname{Int} S$.*

(3) We use "singular arc bc" to denote a map f of the unit interval $(0, 1)$, where $f(0)=b$ and $f(1)=c$.

Proof. We will use Lemmas 3 and 4 to show that $\operatorname{Int} S$ is 1-ULC, and it will then follow from two of Bing's theorems [6, Theorem 2.1], [7, Theorem 1] that $S + \operatorname{Int} S$ is a topological 3-cell and hence that S is tame from $\operatorname{Int} S$.

Let K be a disk and let ϵ be a positive number. By Lemma 3, there exists a $\delta > 0$ such that each map of $\operatorname{Bd} K$ into a δ-subset of $\operatorname{Int} S$ can be extended to map K into an $\epsilon/3$-subset of $\operatorname{Int} S + S - J$. Let f be a map of $\operatorname{Bd} K$ into a δ-subset of $\operatorname{Int} S$. Extend f to map K into an $\epsilon/3$-subset of $\operatorname{Int} S + S - J$. By hypothesis, $\operatorname{Int} S$ is locally simply connected at each point of $S \cdot f(K)$, so it follows from Lemma 4 that $f(\operatorname{Bd} K)$ can be shrunk to a point in $\operatorname{Int} S$ in an $\epsilon/3$-neighborhood of $f(K)$. Hence $f(\operatorname{Bd} K)$ can be shrunk to a point in an ϵ-subset of $\operatorname{Int} S$, so $\operatorname{Int} S$ is 1-ULC.

REMARK. Moise [21] has shown that a 2-sphere in E^3 is tame if it is the sum of two tame disks D_1 and D_2 such that $D_1 \cdot D_2 = \operatorname{Bd} D_1 = \operatorname{Bd} D_2$. This suggests the following theorem which is a corollary to Theorem 2.

THEOREM 3. *If S is a 2-sphere in E^3, J is a tame simple closed curve on S, and S is locally tame from* $\operatorname{Int} S$ *at each point of $S - J$, then S is tame from* $\operatorname{Int} S$.

THEOREM 4. *If M is a connected 2-manifold in E^3 and V is a complementary domain of M such that V is locally simply connected at each point of M, then M is tame from V.*

Proof. Let p be a point of M. Let D be a disk and let S be a 2-sphere which satisfy the requirements of the conclusion of Theorem 1. Now from the fact that V is locally simply connected at each point of M, it follows that $\operatorname{Int} S$ is locally simply connected at each point of $\operatorname{Int} D$. Also, S being locally polyhedral at each point of $S - D$, $\operatorname{Int} S$ is locally simply connected at each point of $S - \operatorname{Bd} D$. Hence, by Theorem 2, S is tame from $\operatorname{Int} S$. From this it follows that M is locally tame at p from V. Now we have shown that M is locally tame from V, and this implies that M is tame from V.

THEOREM 5. *If the complementary domain V of the connected 2-manifold M in E^3 is 1-ULC, then M is tame from V.*

Proof. The hypothesis that V is 1-ULC implies that V is locally simply connected at each point of M. Hence it follows from Theorem 4 that M is tame from V.

THEOREM 6. *If the connected 2-manifold M in E^3 can be homeomorphically approximated from the complementary domain V of M, then M is tame from V.*

Proof. We will show that V is locally simply connected at each point of M, and thus it will follow from Theorem 4 that M is tame from V.

Let p be a point of M and let N be a neighborhood of p. There exists a disk K on M such that $p \in \operatorname{Int} K$ and $K \subset N$. There is a spherical neighborhood U of p such that $U \subset N$ and $U \cdot M \subset K$. Now let f be a map of $\operatorname{Bd} K$ into $U \cdot V$, and extend f to map K into U. By hypothesis, there is a homeomorphism h of M into V such that, $h(K) \subset N$ and $h(M) \cdot f(K) \subset h(K)$. Furthermore, h can be chosen so that $h(K)$ separates $f(K) \cdot M$ from $f(\operatorname{Bd} K)$ in $f(K)$. By Lemma 1, there exists a map g of K into E^3 such that

(1) $g | \operatorname{Bd} K = f | \operatorname{Bd} K$,

(2) $g(K) \subset f(K) + h(K)$, and

(3) $g(K) - h(K)$ is connected.

It follows that $g(K) \subset N \cdot V$, so we have shown that any map of $\operatorname{Bd} K$ into $U \cdot V$ can be shrunk to a point in $N \cdot V$. Thus V is locally simply connected at each point of M.

3. **Surfaces with local spanning disks.** Let M be a connected 2-manifold in E^3, let V be a complementary domain of M, and let p be a point of M. We define M to be *locally spanned at p from V* if for each $\epsilon > 0$, there exist disks R and D such that

(1) $p \in \operatorname{Int} R \subset R \subset M$,

(2) $\operatorname{Int} D \subset V$,

(3) $\operatorname{Bd} D = \operatorname{Bd} R$, and

(4) $\operatorname{diam}(R + D) < \epsilon$.

We say that M is *locally spanned from V* if M is locally spanned from V at each point of M. We say that M can be *locally spanned* if it can be locally spanned from each of its complementary domains.

THEOREM 7. *If the connected 2-manifold M in E^3 can be locally spanned from the complementary domain V of M, then M is tame from V.*

Proof. We will show that V is locally simply connected at each point of M, so that it will follow from Theorem 4 that M is tame from V.

Let p be a point of M and let N be a neighborhood of p. There exists a disk K such that

$$p \in \operatorname{Int} K \subset K \subset N \cdot M. \tag{1}$$

Let U be a spherical neighborhood of p such that

$$U \subset N \quad \text{and} \quad U \cdot M \subset \operatorname{Int} K. \tag{2}$$

Let f be a map of $\operatorname{Bd} K$ into $U \cdot V$, and extend f to a map of K into U. It follows from (2) and Lemma 1 that there is a map g_0 of K into $U \cdot V + \operatorname{Int} K$ such that

$$g_0 | \operatorname{Bd} K = f | \operatorname{Bd} K, \tag{3}$$

(4) $$g_0(K) \subset f(K) + \operatorname{Int} K,$$

and

(5) $$g_0(K) - K \text{ is a connected subset of } U \cdot V.$$

There exists an arc X with end points b and q such that $b \in g_0(\operatorname{Bd} K)$, $q \in M - K$, and $X - q \subset V$. From the hypothesis that M can be locally spanned from V, it follows that there exist disks $R_1, R_2, \cdots, R_m$ and disks $D_1, D_2, \cdots, D_m$ such that

(6) $$g_0(K) \cdot M \subset \sum_{i=1}^{m} \operatorname{Int} R_i$$

and for each i $(1 \leqq i \leqq m)$,

(7) $$R_i \subset K,$$

(8) $$D_i \cdot R_i = \operatorname{Bd} D_i = \operatorname{Bd} R_i,$$

(9) $$\operatorname{Int} D_i \subset N \cdot V,$$

and

(10) $$g_0(\operatorname{Bd} K) + X \subset \operatorname{Ext}(R_i + D_i).$$

Now we will define a finite sequence of maps $g_0, g_1, \cdots, g_m$ of K into $V + M$ so that $g_m | \operatorname{Bd} K = f | \operatorname{Bd} K$ and $g_m(K) \subset N \cdot V$. We define g_i $(1 \leqq i \leqq m)$ by induction on i as follows. Assume that g_{i-1} has been defined. By Lemma 1, there exists a map g_i of K into $V + M$ such that

(11) $$g_i | \operatorname{Bd} K = g_{i-1} | \operatorname{Bd} K,$$

(12) $$g_i(K) \subset g_{i-1}(K) + \operatorname{Int} D_i,$$

and

(13) $$g_i(K) - D_i \text{ is connected.}$$

It follows from (12) that

(14) $$g_i(K) \cdot M \subset g_{i-1}(K) \cdot M.$$

We wish now to show that for each i $(1 \leqq i \leqq m)$, $g_i(K)$ does not intersect $\operatorname{Int} R_i$, so suppose that for some integer j $(1 \leqq j \leqq m)$, $g_j(K)$ intersects $\operatorname{Int} R_j$. It follows that the connected set $g_j(K) - D_j$ intersects $\operatorname{Int} R_j$. Hence $g_j(K) - D_j$ contains an arc Y from the point b in $g_0(\operatorname{Bd} K)$ to a point d in $\operatorname{Int} R_j$ such that $Y - d \subset V + M - (R_j + D_j)$. Let Z be an arc from d to q such that $Z - (d + q) \subset E^3 - (M + V)$. It follows from (10) that $X + Y + Z$ contains a simple closed curve which pierces the disk R_j at d and does not intersect the 2-sphere $D_j + R_j$ at any point different from d. Thus we have a contradiction, so for each i $(1 \leqq i \leqq m)$, $g_i(K)$ does not intersect $\operatorname{Int} R_i$.

Suppose now that there is a point w in $M \cdot g_m(K)$. It follows from (6) and (14) that there is an integer h $(1 \leqq h \leqq m)$ such that $w \in \operatorname{Int} R_h$ and $w \in g_h(K)$. This is contrary to what was shown in the paragraph immediately above. Thus

$$g_m(K) \subset V. \tag{15}$$

It follows from (12) that

$$g_m(K) \subset g_0(K) + \sum_{i=1}^{m} \operatorname{Int} D_i. \tag{16}$$

By combining (2), (5), (9), (15), and (16), we conclude that

$$g_m(K) \subset N \cdot V. \tag{17}$$

Hence we have shown that V is locally simply connected at each point of M.

THEOREM 8. *A connected 2-manifold in E^3 is tame if it can be locally spanned.*

This theorem is a corollary to Theorem 7.

THEOREM 9. *If S is a 2-sphere in E^3, J is a tame simple closed curve on S, and S can be locally spanned from* $\operatorname{Int} S$ *at each point of $S - J$, then S is tame from* $\operatorname{Int} S$.

Proof. An argument similar to the one given for Theorem 7 can be followed to show that $\operatorname{Int} S$ is locally simply connected at each point of $S - J$. Hence it follows from Theorem 2 that S is tame from $\operatorname{Int} S$.

THEOREM 10. *If M is a connected 2-manifold in E^3, V is a complementary domain of M, U is a subset of M that is open relative to M, and M can be locally spanned from V at each point of U, then M is locally tame from V at each point of U.*

Proof. Let p be a point of U, let ϵ be a positive number, and let S and D be a 2-sphere and a disk, respectively, which satisfy the requirements of the conclusion of Theorem 1 such that $D \subset U$. Since S is tame at each point of $S - D$ and can be locally spanned from $\operatorname{Int} S$ at each point of $\operatorname{Int} D$, it follows from Theorem 9 that S is tame from $\operatorname{Int} S$. Hence M is locally tame from V at the point p, so that M is locally tame from V at each point of U.

THEOREM 11. *If M is a connected 2-manifold in E^3, U is a subset of M that is open relative to M, and M can be locally spanned at each point of U from each complementary domain of M, then M is locally tame at each point of U.*

This theorem is a corollary to Theorem 10.

REMARK. Suppose the definition that a connected 2-manifold M can be locally spanned at a point p from the complementary domain V of M is changed so that for each $\epsilon > 0$ there exists a disk R and a map f of R into $M + V$ such that

(1) $p \in \text{Int}\, R \subset R \subset M$,

(2) $f(\text{Int}\, R) \subset V$,

(3) f is the identity on $\text{Bd}\, R$ and is a homeomorphism on some annulus that contains $\text{Bd}\, R$, and

(4) $\text{diam}\,(R + f(R)) < \epsilon$.

Dehn's Lemma, as proved by Papakyriakopoulos [22] and adjusted for nonpiecewise linear maps using Bing's approximation theorem [4, Theorem 7], implies that M can be locally spanned at p from V, so all of the theorems in this section would hold true with the above change. However, suppose we omit the requirement that f is a homeomorphism on some annulus that contains $\text{Bd}\, R$. It is not known whether this latter change implies that M can be locally spanned at p from V. A question about such a strengthened form of Dehn's Lemma has been raised by Bing [12].

4. **Surfaces which can be uniformly locally spanned in a complementary domain.** Let M be a connected 2-manifold in E^3 and let V be a complementary domain of M. We will say that M can be *uniformly locally spanned in* V if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for any δ-disk R on M and for any positive number α, V contains a disk D with the following properties:

(1) $\text{diam}\, D < \epsilon$,

(2) there is a homeomorphism of $\text{Bd}\, R$ onto $\text{Bd}\, D$ that moves no point more than a distance α.

If U is a subset of M that is open relative to M and the above properties hold true for any δ-disk R in U, then we will say that U can be *uniformly locally spanned in* V.

A stronger form of the following lemma can be obtained from the results in a recent paper by Bing [11], but we prove it here in a form that is needed in the proofs of the theorems in this section.

LEMMA 5. *Suppose S is a 2-sphere in E^3, K is a disk on S, and f is a map of K into* $\text{Int}\, S + \text{Int}\, K$ *such that* $f(\text{Bd}\, K) \subset \text{Int}\, S$. *Then for each positive number δ there exists a map g of K into* $\text{Int}\, S + \text{Int}\, K$ *such that*

(i) $g|\text{Bd}\, K = f|\text{Bd}\, K$,

(ii) *every point of $g(K)$ is within a distance δ of $f(K) + K$, and*

(iii) *$g(K) \cdot S$ can be covered by a finite number of disjoint δ-disks in* $\text{Int}\, K$.

Proof. As in the proof of Theorem 1, there exists a subdisk K' of K such that $f(K) \cdot K \subset \text{Int}\, K'$ and $\text{Bd}\, K'$ is tame. Hence for convenience we will assume that $K = K'$ so that $\operatorname{Bd} K$ is tame. There exists a disk K_1 in $\operatorname{Int} K$ such that $f(K) \cdot K \subset \operatorname{Int} K_1$. Let δ be a positive number such that

$$\delta < \min[\rho(S - K, K_1), \rho(S - K_1, f(K))], \tag{1}$$

where ρ denotes the distance function. It follows from Bing's Side Approximation Theorem [10] that there exist a finite number of disjoint disks $D_1, D_2, \cdots, D_m$ on S, a polyhedral 2-sphere S', and a homeomorphism h of S onto S' such that

(2) h moves no point more than a distance $\delta/2$,

$$S - \sum_{i=1}^{m} D_i \subset \operatorname{Ext} S', \tag{3}$$

and

$$\text{for each } i, \operatorname{diam} D_i < \delta/2. \tag{4}$$

Furthermore, we can require [20, p. 97] that

$$f(\operatorname{Bd} K) \subset \operatorname{Int} S' \tag{5}$$

and, as shown by Gillman [16, p. 464], that

$$\operatorname{Bd} K \text{ does not intersect } \sum_{i=1}^{m} D_i. \tag{6}$$

It follows from (1) and (2) that

$$f(K) \cdot S' \subset h(K_1) \quad \text{and} \quad h(K_1) \cdot S \subset \operatorname{Int} K. \tag{7}$$

By (5), (7), and Lemma 1, there exists a map f' of K into $\operatorname{Int} S' + h(K_1)$ such that

$$f' | \operatorname{Bd} K = f | \operatorname{Bd} K \tag{8}$$

and

$$f'(K) \subset f(K) + h(K_1). \tag{9}$$

Now it follows from (2) and (9) that

(10) every point of $f'(K)$ is within a distance $\delta/2$ of $f(K) + K$.

For convenience, let $D_1, D_2, \cdots, D_n$ denote the D_i's that intersect K. By (3), (6), and (7),

$$S \cdot f'(K) \subset \sum_{i=1}^{n} D_i \subset \operatorname{Int} K.$$

Hence Lemma 1 implies that there exists a map g defined on K such that

$$g | \operatorname{Bd} K = f' | \operatorname{Bd} K = f | \operatorname{Bd} K, \tag{11}$$

(12) $$g(K) \subset \operatorname{Int} S + \sum_{i=1}^{n} D_i,$$

and

(13) $$g(K) \subset f'(K) + \sum_{i=1}^{n} D_i.$$

It follows from (4) and (13) that

(14) every point of $g(K)$ is within a distance $\delta/2$ of $f'(K)$.

Hence, by combining (10) and (14), we conclude that

(15) every point of $g(K)$ is within a distance δ of $f(K) + K$.

Now the requirements in the conclusion of Lemma 5 follow from (11), (12), and (15).

THEOREM 12. *If S is a 2-sphere in E^3 and H is a disk on S such that* $\operatorname{Int} H$ *can be uniformly locally spanned in* $\operatorname{Int} S$, *then* $\operatorname{Int} S$ *is locally simply connected at each point of* $\operatorname{Int} H$.

Proof. Let p be a point of $\operatorname{Int} H$ and let N be a neighborhood of p. There exists a subdisk K of H such that

(1) $$p \in \operatorname{Int} K \quad \text{and} \quad K \subset N.$$

Let U be a spherical neighborhood of p such that

(2) $$U \subset N \quad \text{and} \quad U \cdot S = U \cdot K.$$

Let f be a map of $\operatorname{Bd} K$ into $U \cdot \operatorname{Int} S$. We will show that $f(\operatorname{Bd} K)$ can be shrunk to a point in $N \cdot \operatorname{Int} S$, and this will imply that $\operatorname{Int} S$ is locally simply connected at each point of $\operatorname{Int} H$.

First extend f to map K into U. Then apply Lemma 1 to obtain a map f' of K into $K + \operatorname{Int} S$ such that

(3) $$f' | \operatorname{Bd} K = f | \operatorname{Bd} K,$$

(4) $$f'(K) \subset f(K) + \operatorname{Int} K,$$

and

(5) $$f'(K) - K \text{ is connected.}$$

It follows from (2), (4), and (5) that

(6) $$f'(K) - K \subset N \cdot \operatorname{Int} S.$$

There exists an arc X with end points b and q such that $b \in f(\operatorname{Bd} K)$, $q \in S - K$, and $X - q \subset \operatorname{Int} S$. Let ϵ be a positive number such that

(7) $$2\epsilon < \rho(f'(K) + K, \operatorname{Bd} N) \quad \text{and} \quad 2\epsilon < \rho(K, X + f(\operatorname{Bd} K)).$$

Let δ be a positive number satisfying the requirements, relative to ϵ defined in (7), that are implied by the hypothesis that $\text{Int}\,H$ can be uniformly locally spanned in $\text{Int}\,S$. Also, require that

(8) $$2\delta < \epsilon.$$

Apply Lemma 5 to get a map g_0 of K into $\text{Int}\,S + \text{Int}\,K$ such that

(9) $$g_0|\text{Bd}\,K = f'|\text{Bd}\,K = f|\text{Bd}\,K,$$

(10) every point of $g_0(K)$ is within a distance δ of $f'(K) + K$, and

(11) $g_0(K) \cdot S$ can be covered by the interiors of a finite number of disjoint δ-disks $R_1, R_2, \cdots, R_n$ in $\text{Int}\,K$.

For each i $(1 \leqq i \leqq n)$, let R_i' be a disk in $\text{Int}\,R_i$ such that $R_i \cdot g_0(K) \subset R_i'$. Let c_i be a point of R_i', and let Z_i be an arc from c_i to q such that $Z_i - (c_i + q) \subset \text{Ext}\,S$.

Now by use of Lemma 1 and induction on i, we can define a finite sequence of positive numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, a finite sequence of disks $D_1, D_2, \cdots, D_n$, and a finite sequence of maps $g_0, g_1, \cdots, g_n$ of K into $N \cdot \text{Int}\,S + K$ such that for each i $(1 \leqq i \leqq n)$,

(12) $$\alpha_i < \epsilon,$$

(13) $$\alpha_i < \rho(\text{Bd}\,R_i, R_i' + g_{i-1}(K) + Z_i + X),$$

(14) $$D_i \subset \text{Int}\,S,$$

(15) $$\text{diam}\,D_i < \epsilon,$$

(16) there is a homeomorphism of $\text{Bd}\,R_i$ onto $\text{Bd}\,D_i$ that moves no point more than a distance α_i,

(17) $$g_i|\text{Bd}\,K = g_{i-1}|\text{Bd}\,K,$$

(18) $$g_i(K) \subset g_{i-1}(K) + \text{Int}\,D_i,$$

and

(19) $$g_i(K) - D_i \text{ is connected.}$$

We wish to show that for each i $(1 \leqq i \leqq n)$, $g_i(K)$ does not intersect R_i, so suppose that for some integer j $(1 \leqq j \leqq n)$, $g_j(K)$ intersects R_j. It follows from (11) and (14) that $g_j(K) - D_j$ intersects $\text{Int}\,R_j$. Since $R_j \cdot g_j(K) \subset R_j'$, it follows that the connected set $g_j(K) - D_j$ contains an arc Y_j from the point b in $g_0(\text{Bd}\,K)$ to a point d_j in R_j' such that

(20) $$Y_j - d_j \subset \text{Int}\,S + S - (R_j + D_j).$$

Furthermore, (18) and (20) imply that

(21) $$Y_j \subset g_{j-1}(K).$$

Let W_j be an arc in R_j' from c_j to d_j. Now $X + W_j + Y_j + Z_j$ contains a simple closed curve L which contains the arcs W_j and Z_j and a subarc of

Y_j that contains d_j. Hence L links $\mathrm{Bd}\, R_j$. It follows from (13) and (21) that no point of L is within a distance α_j of $\mathrm{Bd}\, R_j$. Hence it follows from (16) and [4, Theorem 10] that L links $\mathrm{Bd}\, D_j$. But this is impossible as (7), (12), (14), (15), (16) and (20) imply that L does not intersect D_j. Hence for each i $(1 \leqq i \leqq n)$, $g_i(K)$ does not intersect R_i. As in the proof of Theorem 7, it follows that $g_n(K)$ does not intersect $\sum_{i=1}^{n} R_i$, so that

$$g_n(K) \subset \mathrm{Int}\, S. \tag{22}$$

Now we need to show that $g_n(K) \subset N$. From (1), (6), (7), (8), and (10), we conclude that

$$g_0(K) \subset N. \tag{23}$$

It follows from (12), (15), and (16) that for each i $(1 \leqq i \leqq n)$, every point of D_i is within a distance 2ϵ of R_i and hence is within a distance 2ϵ of K. Thus by (7), it follows that

$$\sum_{i=1}^{n} D_i \subset N. \tag{24}$$

It is a consequence of (18) that

$$g_n(K) \subset g_0(K) + \sum_{i=1}^{n} D_i. \tag{25}$$

Now (23), (24), and (25) imply that $g_n(K) \subset N$. Hence we have shown that $f(\mathrm{Bd}\, K)$ can be shrunk to a point in $N \cdot \mathrm{Int}\, S$.

THEOREM 13. *If M is a connected 2-manifold in E^3, V is a complementary domain of M, and M can be uniformly locally spanned in V, then M is tame from V.*

Proof. Let p be a point of M. Let S and D be a 2-sphere and a disk, respectively, which satisfy the last four requirements in the conclusion of Theorem 1. Let D' be a disk in $\mathrm{Int}\, D$ such that $p \in \mathrm{Int}\, D'$. Now the hypothesis that M can be uniformly locally spanned in V implies that $\mathrm{Int}\, D'$ can be uniformly locally spanned in $\mathrm{Int}\, S$. By Theorem 12, $\mathrm{Int}\, S$ is locally simply connected at p, and this implies that V is locally simply connected at p. Hence V is locally simply connected at each point of M, so that it follows from Theorem 4 that M is tame from V.

REMARK. We observe that theorems similar to Theorems 8, 10, and 11 can be proved for connected 2-manifolds which have the uniform spanning property defined at the beginning of §4. Also we observe that the same arguments could be used for Theorems 12 and 13 if requirement (2) in this definition were changed to require that the identity map on $\mathrm{Bd}\, R$ can be shrunk to a point in $D +$ (some α-neighborhood of $\mathrm{Bd}\, R$).

5. **Surfaces with certain types of subsets that can be collared.** Let M be a 2-manifold in E^3, let V be a complementary domain of M, and let I denote the unit interval $(0, 1)$. We modify Brown's definition [13] to say that a subset H of M can be *collared from* V if the Cartesian product $H \times I$ can be imbedded in E^3 so that

(1) $b \times 0 = b$, where $b \in H$, and

(2) $b \times t \in V$, where $0 < t \leqq 1$.

Saying that each point of M lies in an open set on M that can be collared from V would be equivalent to saying, under Brown's definition, that M is locally collared in $M + V$.

Bing has asked whether a 2-sphere S in E^3 is tame if every universal plane curve can be pushed into each complementary domain of S with a homotopy [12]. We do not answer this question, but we apply Theorem 7 to give an affirmative answer for the case where every such curve can be collared from each complementary domain of S.

A *universal plane curve,* sometimes called a Sierpiński curve, is a continuum H obtained by removing the interiors of a sequence of disjoint disks $D_1, D_2, \cdots$ from a 2-sphere S, where $\sum_{i=1}^{\infty} D_i$ is dense in S and the diameters of $D_1, D_2, \cdots$ converge to zero [23]. Any point of H that is not in $\sum_{i=1}^{\infty} \operatorname{Bd} D_i$ is called an *inaccessible point* of H.

THEOREM 14. *Let M be a connected 2-manifold in E^3 and let V be a complementary domain of M. Suppose that for each point p of M, there is a sequence of disks $R_1, R_2, \cdots$ on M such that*

(1) for $i > 1$, $p \in \operatorname{Int} R_i$ and $R_i \subset \operatorname{Int} R_{i-1}$,

(2) p is the intersection of $R_1, R_2, \cdots$, and

(3) $p + \sum_{i=1}^{\infty} \operatorname{Bd} R_i$ can be collared from V.

Then M is tame from V.

Proof. We will show that M can be locally spanned from V, so it will follow from Theorem 7 that M is tame from V.

Let p be a point of M, let ϵ be a positive number, and let $R_1, R_2, \cdots$ be a sequence of disks satisfying the requirements in the hypothesis of Theorem 14 such that $\operatorname{diam} R_1 < \epsilon/6$. Since the set $p + \sum_{i=1}^{\infty} \operatorname{Bd} R_i$ can be collared from V, it follows that there is an imbedding of $(p + \sum_{i=1}^{\infty} \operatorname{Bd} R_i) \times I$ in an $\epsilon/3$-subset of $M + V$ such that

$$(1) \qquad b \times 0 = b, \qquad \text{where } b \in p + \sum_{i=1}^{\infty} \operatorname{Bd} R_i,$$

and

$$(2) \qquad b \times t \in V, \qquad \text{where } 0 < t \leqq 1.$$

Let S be a polyhedral 2-sphere such that

$$(3) \qquad \operatorname{diam} S < \epsilon/3,$$

$$p \times 1 \in \operatorname{Int} S, \tag{4}$$

and

$$S \subset V. \tag{5}$$

There exists an integer j such that $\operatorname{Bd} R_j \times 1 \subset \operatorname{Int} S$. Using Bing's approximation theorem [4], we can require that $\operatorname{Bd} R_j \times I$ is locally polyhedral except on $\operatorname{Bd} R_j$ and that each component of $S \cdot (\operatorname{Bd} R_j \times I)$ is a simple closed curve. There exists a disk D_j such that

$$D_j \subset (\operatorname{Bd} R_j \times I) + S, \tag{6}$$

$$\operatorname{Bd} D_j = \operatorname{Bd} R_j, \tag{7}$$

and

$$\operatorname{diam}(D_j + R_j) < \epsilon. \tag{8}$$

Thus M can be locally spanned at p from V, so it follows that M can be locally spanned from V.

THEOREM 15. *If M is a connected 2-manifold in E^3 and V is a complementary domain of M such that every point of M is an inaccessible point of some universal plane curve on M that can be collared from V, then M is tame from V.*

Proof. Let p be a point of M and let H be a universal plane curve such that p is an inaccessible point of H and H can be collared from V. There exists a sequence of disks $R_1, R_2, \cdots$ satisfying the requirements in the hypothesis of Theorem 14 such that for each i, $\operatorname{Bd} R_i \subset H$. Thus it follows from Theorem 14 that M is tame from V.

6. **Generalizations to 2-manifolds in a 3-manifold.** Let M^2 denote a 2-manifold in a 3-manifold M^3 and let p denote a point of M^2. Using [7, Theorem 5], we observe that there is a 2-sphere S such that p is in the interior of a disk D in $S \cdot M^2$ and S is a subset of a neighborhood N of p that is homeomorphic to E^3. We say that M^2 can be *locally spanned* at p if some such sphere S can be locally spanned at p from each component of $N - S$. Notice that this is equivalent to requiring that each such sphere S can be locally spanned at p from each component of $M^3 - S$. Similarly, we modify the definitions in §4 to say that M^2 can be *uniformly locally spanned* in $M^3 - M^2$ if for each point p of M^2 and some such sphere S, $\operatorname{Int} D$ can be uniformly locally spanned in each component of $N - S$.

THEOREM 16. *If M^2 is a 2-manifold in a triangulated 3-manifold M^3 and M^2 can be locally spanned at each of its points, then M^2 is tame in M^3.*

Proof. Let p be a point of M^2 and let N be a neighborhood of p that is homeomorphic to E^3. We assume that N has a triangulation that is derived from the given triangulation of M^3. By the above definition, there exists a disk D and a 2-sphere S such that

$$p \in \operatorname{Int} D \subset D \subset S \cdot M^2 \subset S \subset N$$

and S can be locally spanned at each point of $\operatorname{Int} D$ from each component of $N - S$. Hence by Theorem 11, S is locally tame in N at each point of $\operatorname{Int} D$. This implies that M^2 is locally tame in M^3 at each point of $\operatorname{Int} D$, so it follows that M^2 is tame in M^3 [3], [21].

THEOREM 17. *If M^2 is a 2-manifold in a triangulated 3-manifold M^3, U is a subset of M^2 that is open relative to M^2, and M^2 can be locally spanned at each point of U, then M^2 is locally tame at each point of U.*

Proof. This theorem follows from the proof of Theorem 16 if it is required that $N \cdot M^2 \subset U$.

THEOREM 18. *If M^2 is a 2-manifold in a 3-manifold M^3 and M^2 can be uniformly locally spanned in $M^3 - M^2$, then M^2 is tame in M^3.*

Proof. Choose p, N, S, and D as in the proof of Theorem 16 so that $\operatorname{Int} D$ can be uniformly locally spanned in each component of $N - S$. We observe that Theorem 12 holds true where $\operatorname{Int} S$ is replaced with $\operatorname{Ext} S$. Hence it follows from Theorem 12 that $N - S$ is locally simply connected at each point of $\operatorname{Int} D$. Thus $M^3 - M^2$ is locally simply connected at each point of M^2, and Bing [7, Theorem 7] has shown that this implies that M^2 is tame in M^3.

REMARK. We observe that by similar methods, Theorems 14, 15, and 18 can be generalized to give conditions under which a 2-manifold M^2 in a 3-manifold M^3 is locally tame at each point of an open subset of M^2.

Added in proof. L. D. Loveland [Notices Amer. Math. Soc. **11** **(1964)**, 313] has recently proved Theorem 13 without the requirement of uniformity provided it is required that M can be locally spanned in V at each point of M relative to a disk on M with a tame boundary. A similar change can be made in Theorems 12 and 18.

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